

Effective betting with restricted wagers

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Abstract

Computable randomness is a central notion in the theory of algorithmic randomness. An infinite sequence of bits x is computably random if no computable betting strategy can win an infinite amount of money by betting on the values of the bits of x . In the classical model, the betting strategies considered take real-valued bets. We study two restricted models, where the strategies considered take bets in subsets of the real numbers. The subsets are (a) the integers; and (b) the real numbers excluding a punctured neighborhood of 0. We consider, also, two alternative tests for randomness: no computable betting strategy can (i) save an infinite amount of money; (ii) oscillate by betting on the values of the bits of x . The alternative tests are equivalent to the original test in the classic model, but they turn out to be different in the restricted models.

Our results include solving questions raised in Bienvenu et al. (2012); Teutsch (2012), strengthening existing results, and extending the theory to higher Turing degrees.

1 Introduction

1.1 The casino setting

A sequence of players enter the casino. Player 1 declares her betting strategy, a function from finite histories of Heads and Tails to real-valued bets. Then the rest of the players 2,3,... (countably many of them) declare their strategies, but they are restricted to integer-valued bets. The casino wants Player 1 to win while all the others lose. That is, the casino should choose a sequence of Heads and Tails so that the limit of Player 1's capital is infinite while all others' is finite.

Is it possible? We show it is. Consider the following strategy for Player 1. She enters the casino with an irrational amount of money x_0 dollars. After t periods she has x_t dollar in her pocket and she bets $\frac{1}{2} \{x_t\}$ on Heads (where $\{x\} := x - \lfloor x \rfloor$).

Now, Players 2,3,... declare their betting strategies. The casino places a finite sequence of Heads and Tails σ on which the capital of player 2 is minimal. Recall, Players 2,3,... may only bet integer numbers; hence that minimum exists. At this point Player 2 is ruined. Should he ever place a (non-zero) bet, that would contradict the fact that σ is a

minimizer of his capital. Note that since Player 1 only bets on the fractional part of her capital, $\lfloor x_t \rfloor$ never decreases.

In the next stage, the casino extends σ by appending to it sufficiently many Heads, so that Player 1's capital increases by at least 1. In turn, the casino repeats the same trick against every player, ruining him while Player 1 does not lose more than the fractional part of her capital, and placing Heads until she accumulates a dollar. QED.

1.2 Computable randomness

Our motivation comes from the study of computable randomness. We give here a short account of computed randomness. The reader may refer to Nies (2009) for more details.

Computable randomness, as introduced by Schnorr (1971), and its relative variant classify binary sequences in terms of the oracles needed in order to effectively predict the sequence. A binary sequence x is computably random (relative to an oracle O), if there is no (O -)computable betting strategy that accumulates an infinite amount of money by playing against the bits of x .

We are motivated by refinements of the notion of (relative) computable randomness recently introduced by Bienvenu et al. (2012); Teutsch (2012). A predictability class is defined by a set of permissible betting strategies and a success criterion. A binary sequence is called predictable (not random) with respect to that class (relative to an oracle O), if some (O -)computable permissible betting strategy achieves the success criterion on x .

We call the success criterion of the main notion of computable randomness ∞ -GAINS. Other, well studied, success criteria include those of Schnorr (1971) and Kurtz (1981). A few, less familiar, success criteria turned out to be equivalent to the main notion of computable randomness and became folklore. When Bienvenu et al. (2012) introduced integer-valued betting strategies some of the folklore criteria gained renewed interest. Teutsch (2012) studies a success criterion we call ∞ -SAVINGS with respect to integer-valued betting strategies.

Both Bienvenu et al. (2012) and Teutsch (2012) raise questions as to whether or not their results extend to betting strategies that take bets in the set $V = \{x \in \mathbb{R} : |x| \geq 1 \text{ or } x = 0\}$. We answer those questions and we investigate a further success criterion, OSCILLATION, in which the gambler's capital needs to oscillate. We provide a complete characterization of the relations between nine refinements of the notion of computable randomness. The refinements are given by three sets of permissible betting, \mathbb{R} , V and \mathbb{Z} , and three success criteria ∞ -GAINS, ∞ -SAVINGS and OSCILLATION. We find that these refinements define five different notions which are linearly ordered, one implying the next one. Figure 1.2 summarizes these relations.

The casino setting described above facilitates the separation between these refinements. The example in the previous section corresponds to separating $(\infty\text{-GAINS}, \mathbb{R})$ from $(\infty\text{-GAINS}, \mathbb{Z})$. Our separation results are robust in the following sense: the separating strategy (of Player 1) is computable, history independent, and independent of the strategies for the other players (regardless of their computation power); in all but one case, we manage to show a separating sequence (of the casino) uniformly computable relative to an enumeration of the strategies of the players.

2 Definitions and Results

2.1 Definitions

The set all finite bit strings is denoted $\{-1, +1\}^{<\infty} = \bigcup_{n=0}^{\infty} \{-1, +1\}^n$. The length of a string $\sigma \in \{-1, +1\}^{<\infty}$ is denoted $|\sigma|$. The empty string is denoted ε . For an infinite bit sequence $x \in \{-1, +1\}^{\mathbb{N}}$ and a non-negative integer n , the prefix of x of length n is denoted by $x \upharpoonright n$.

A *betting strategy* is a function $M : \{-1, +1\}^{<\infty} \rightarrow \mathbb{R}$, satisfying the (super) fairness condition

$$M(\sigma) \geq \frac{M(\sigma, -1) + M(\sigma, +1)}{2},$$

for every $\sigma \in \{-1, +1\}^{<\infty}$. If an equality holds for every $\sigma \in \{-1, +1\}^{<\infty}$, we say that M is *balanced*.

The *wager* of M at σ is defined as

$$M'(\sigma) = \frac{M(\sigma, +1) - M(\sigma, -1)}{2}.$$

Note that $M'(\sigma)$ is positive if M wagers on “+1” and negative if M wagers on “−1” at σ . The *initial capital* of M is defined as $M(\varepsilon)$. A balanced betting strategy is determined by its initial capital and its wagers at every $\sigma \in \{-1, +1\}^{<\infty}$.

For a betting strategy M , the *balancing* of M is the balanced betting strategy, \tilde{M} , whose initial capital and wagers are the same as M ’s.

A betting strategy is called *history-independent* if $M'(\sigma) = M'(\tau)$, whenever $|\sigma| = |\tau|$.

For a betting strategy M and a string σ , we say that M *goes bankrupt at* σ , if

$$M(\sigma) < |M'(\sigma)|.$$

For an infinite sequence $x \in \{-1, +1\}^{\mathbb{N}}$, we say that M *goes bankrupt on* x , if M goes bankrupt at $x \upharpoonright n$, for some non-negative integer n .

Let M be a betting strategy and $x \in \{-1, +1\}^{\mathbb{N}}$. If M does not go bankrupt on x , we say that M *acheives*

- ∞ -GAINS on x , if $\lim_{n \rightarrow \infty} M(x \upharpoonright n) = \infty$;
- ∞ -SAVINGS on x , if $\lim_{n \rightarrow \infty} (\tilde{M} - M)(x \upharpoonright n) = \infty$.

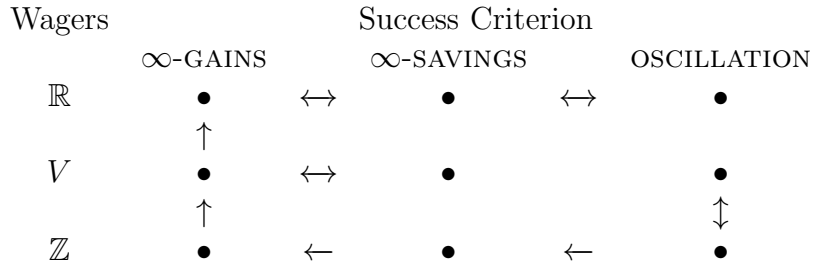


Figure 1: Relations between classes. Arrows indicate implication.

- OSCILLATION on x , if $\liminf_{n \rightarrow \infty} M(x \upharpoonright n) \neq \limsup_{n \rightarrow \infty} M(x \upharpoonright n)$.

We call ∞ -GAINS, ∞ -SAVINGS and OSCILLATION *success criteria*.

For $A \subset \mathbb{R}$, an A -betting strategy is a betting strategy whose wagers take values in A . We will be mainly interested in restricting the wagers to the set of integers \mathbb{Z} and the set $V := \{a \in \mathbb{R} : |a| \geq 1\}$.

We define a *predictability class* (class, in short) as a pair $\mathcal{C} = (A, C)$, where $A \subset \mathbb{R}$ and $C \in \{\infty\text{-GAINS}, \infty\text{-SAVINGS}, \text{OSCILLATION}\}$. We use “ \mathcal{C} -betting strategy achieves \mathcal{C} ” for “ A -betting strategy achieves C .”

Definition 2.1. We say that a class (A_1, C_1) *implies* another class (A_2, C_2) , and write $(A_1, C_1) \longrightarrow (A_2, C_2)$ if for every $x \in \{-1, +1\}^{\mathbb{N}}$ and A_1 -betting strategy M_1 that achieves C_1 on x , there exists an A_2 -betting strategy M_2 such that

- (a) M_2 achieves C_2 on x ; and
- (b) M_2 is computable relative to M_1 .

It turns out that the classes we study exhibit the property that if they do not imply each other, they satisfy a stronger relation that just the negation of implication.

Definition 2.2. We say that a class (A_1, C_1) *anti-implies* another class (A_2, C_2) , and write $(A_1, C_1) \not\rightarrow (A_2, C_2)$, if there exists a computable history-independent A_1 -betting strategy, M_1 , such that for any countable set of A_2 -betting strategies, \mathcal{B} , there exists a sequence $x \in \{-1, +1\}^{\mathbb{N}}$ on which

- (a) M_1 achieves C_1 ;
- (b) none of the elements of \mathcal{B} achieves C_2 ; and
- (c) x is computable relative to an enumeration of \mathcal{B} .

If (a) and (b) hold, but not necessarily (c), we say that (A_1, C_1) *weakly anti-implies* (A_2, C_2) , and write $(A_1, C_1) \xrightarrow{w} (A_2, C_2)$.

Weak anti-implication behaves similar to the negation of implication in the following sense.

Lemma 2.3. Let $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 be classes. If $\mathcal{C}_2 \longrightarrow \mathcal{C}_3$ and $\mathcal{C}_1 \xrightarrow{w} \mathcal{C}_3$, then $\mathcal{C}_1 \xrightarrow{w} \mathcal{C}_2$.

Proof. Take a betting strategy M_1 that separates \mathcal{C}_1 from \mathcal{C}_3 . Let \mathcal{B} be a countable set of \mathcal{C}_2 -betting strategies. Let \mathcal{B}' be the set of all \mathcal{C}_3 -betting strategies computable from an enumeration of \mathcal{B} . There exists a sequence $x \in \{-1, +1\}^{\mathbb{N}}$ on which M_1 achieves \mathcal{C}_1 , but no element of \mathcal{B}' achieves \mathcal{C}_3 . Since $\mathcal{C}_2 \longrightarrow \mathcal{C}_3$, no element of \mathcal{B} achieves \mathcal{C}_2 on x . \square

2.2 Implication results

Lemma 2.4. For every $0 \in A \subset \mathbb{R}$. $(A, \text{OSCILLATION}) \longrightarrow (A, \infty\text{-SAVINGS})$.

Lemma 2.5. $(V, \text{OSCILLATION}) \longrightarrow (\{0, -1, +1\}, \text{OSCILLATION})$

Lemma 2.6. $(\mathbb{R}, \infty\text{-GAINS}) \longrightarrow (\mathbb{R}, \text{OSCILLATION})$

Theorem 2.7. $(V, \infty\text{-GAINS}) \longrightarrow (V, \infty\text{-SAVINGS})$

2.3 Non-implication results

Theorem 2.8 (Teutsch (2012)). $(\{1\}, \infty\text{-GAINS}) \not\rightarrow (\mathbb{Z}, \infty\text{-SAVINGS})$

Theorem 2.9. $(\{1\}, \infty\text{-SAVINGS}) \not\rightarrow (V, \text{OSCILLATION})$

Theorem 2.10. $(\mathbb{R}, \infty\text{-GAINS}) \xrightarrow{w} (V, \infty\text{-GAINS})$

Theorem 2.11. $(V, \infty\text{-GAINS}) \not\rightarrow (\mathbb{Z}, \infty\text{-GAINS})$

3 Proofs

Proof of Lemma 2.4. Let $0 \in A \subset \mathbb{R}$, $x \in \{-1, +1\}^{\mathbb{N}}$, and M an A -betting strategy that oscillates on x . We assume wlog that M is balanced, because if M oscillates so does its balancing \tilde{M} . Take $a, b \in \mathbb{R}$, such that

$$\liminf_{n \rightarrow \infty} M(x \upharpoonright n) < a < b < \limsup_{n \rightarrow \infty} M(x \upharpoonright n).$$

For $y \in \{-1, +1\}^{\mathbb{N}}$, define stopping times $n_0(y), n_1(y), \dots$ recursively by

$$\begin{aligned} n_0(y) &= \inf \{n \geq 0 : M(y \upharpoonright n) < a\}, \\ n_{2i+1}(y) &= \inf \{n > n_{2i} : M(y \upharpoonright n) > b\}, \\ n_{2(i+1)}(y) &= \inf \{n > n_{2i+1} : M(y \upharpoonright n) < a\}, \end{aligned}$$

where the infimum of the empty set is defined as ∞ .

We define an A -betting strategy S by specifying S' , $S(\varepsilon)$ and $f = \tilde{S} - S$ as follows: $S(\varepsilon) = 2a$; before time $n_0(y)$, $S' \equiv 0$ and $f \equiv 0$. For $n_{2i}(y) \leq t < n_{2i+1}(y)$ $S'(y \upharpoonright t) = M'(y \upharpoonright t)$; otherwise $S' = 0$. At times $\{n_{2i+1}(y)\}_{i=0}^{\infty}$, f increases by $b - a$, otherwise f doesn't change. \square

Proof of Lemma 2.5. Let $x \in \{-1, +1\}^{\mathbb{N}}$, and M a V -betting strategy that oscillates on x . We assume wlog that M is balanced. Let $L = \liminf_{n \rightarrow \infty} M(x \upharpoonright n)$. There exists $t_0 \in \mathbb{N}$ such that $M(x \upharpoonright t) > L - \frac{1}{2}$, for every $t \geq t_0$, and $|M(x \upharpoonright t) - L| < \frac{1}{2}$ infinitely often.

The following balanced $\{0, -1, +1\}$ -betting strategy, S , oscillates on x :

$$\begin{aligned} S(\varepsilon) &= 1, \\ S'(y \upharpoonright t) &= \begin{cases} \text{sign}(M'(y \upharpoonright t)) & \text{if } t \geq t_0, |M(x \upharpoonright t) - L| < \frac{1}{2}, \text{ and } S(y \upharpoonright t) = 1, \\ -\text{sign}(M'(y \upharpoonright t)) & \text{if } t \geq t_0, |M(x \upharpoonright t) - L| < \frac{1}{2}, \text{ and } S(y \upharpoonright t) = 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Where $\text{sign}(0) := 0$. \square

Proof of Lemma 2.6. Let $x \in \{-1, +1\}^{\mathbb{N}}$ and M an \mathbb{R} -betting strategy that makes ∞ -GAINS on x . For $y \in \{-1, +1\}^{\mathbb{N}}$, define stopping times $n_0(y), n_1(y), \dots$ recursively by

$$\begin{aligned} n_0(y) &= 0, \\ n_{i+1}(y) &= \inf \{n > n_i : M(y \upharpoonright n) \geq 2M(y \upharpoonright n_i)\}. \end{aligned}$$

Define a balanced betting strategy S by

$$S(\varepsilon) = 5,$$

and for $n_i \leq t < n_{i+1}$,

$$S'(y \upharpoonright t) = \frac{M'(y \upharpoonright t)}{M(y \upharpoonright n_i)} (-1)^{\chi(S(y \upharpoonright n_i) \geq 5)}.$$

□

Proof of Theorem 2.7. Let $x \in \{-1, +1\}^{\mathbb{N}}$ and M a V -betting strategy that makes ∞ -GAINS on x . Assume without loss of generality that M is balanced and $M(\varepsilon) \geq 2$.

For $y \in \{-1, +1\}^{\mathbb{N}}$, define stopping times $n_0(y), n_1(y), \dots$ recursively by

$$\begin{aligned} n_0(y) &= 0, \\ n_{i+1}(y) &= \inf \{n > n_i : M(y \upharpoonright n) \geq 2M(y \upharpoonright n_i)\}. \end{aligned}$$

Define a betting strategy S specified by $S(\varepsilon)$, S' , and $f = \tilde{S} - S$.

$$S(\varepsilon) = 2M(\varepsilon),$$

and for $n_i \leq t < n_{i+1}$,

$$\begin{aligned} S'(y \upharpoonright t) &= (M'(y \upharpoonright t)) \left(2 - \sum_{j < i} \frac{1}{M(y \upharpoonright n_j)}\right), \\ f(y \upharpoonright t) &= i. \end{aligned}$$

□

Proof of Theorem 2.9. Denote by M the balanced $\{1\}$ -betting strategy with initial capital 1. Define a saving function

$$f(\sigma) = \left\lfloor \frac{1}{2} \max_{0 \leq k \leq n} M(\sigma \upharpoonright k) \right\rfloor,$$

and denote

$$m(\sigma) = M(\sigma) - f(\sigma).$$

Note that our definition is so that M, m and f diverge to ∞ on the same set of sequences, and f increases only after two consecutive $+1$'s, and never increases twice in a row.

Let \mathcal{B} be a countable set of balanced (assumed wlog) V -betting strategies. Arrange the elements of $\mathcal{B} \times \mathbb{N}$ is a sequence $\{(S_e, k_e)\}_{e=1}^{\infty}$, such that whenever $e < e'$ and $S_e = S_{e'}$, then $k_e < k_{e'}$.

We define a sequence $x \in \{-1, +1\}^{\mathbb{N}}$ recursively. Assume $x \upharpoonright n$ is defined for some $n \geq 0$. We say that (S_e, k_e) gets attention at time n , if e is minimal with respect to the following properties:

- (i) $S_e(x \upharpoonright n) \leq k_e$,
- (ii) $m(x \upharpoonright n) > e$,
- (iii) $S'_e(x \upharpoonright n) \neq 0$.

Define

$$X_{n+1} = \begin{cases} -\text{sign} S'_e(x \upharpoonright n) & \text{if some } (S_e, k_e) \text{ gets attention at time } n, \\ +1 & \text{otherwise.} \end{cases}$$

Our definition is so that no (S_e, k_e) gets attention at time n when $m(x \upharpoonright n) = 1$; therefore $m(x \upharpoonright t) \geq 1$, for every t .

Let L be an arbitrary integer satisfying $1 \leq L \leq \liminf_{n \rightarrow \infty} m(x \upharpoonright n)$. Since m is integer-valued there exists some $n_0 \in \mathbb{N}$ such that $m(x \upharpoonright n) \geq L$, for every $n \geq n_0$. Consider the set of indexes $I = \{n \geq n_0 : m(x \upharpoonright n) > L\}$. Since f never increases twice in a row, $m(x \upharpoonright n)$ never stabilizes and hence I is infinite.

Consider the sequence of L -tuples of integers $\{a_n = \langle \min\{k_e, \lfloor S_e(x \upharpoonright n) \rfloor\} \rangle_{e=1}^L\}_{n \in I}$. Our definition is so that a_n never increases (with respect to the lexicographic order on \mathbb{Z}^L) and it decreases whenever $m(x \upharpoonright n) - 1 = m(x \upharpoonright n + 1) = L$ and then $m(x \upharpoonright n + 2) = L + 1$, since f increases only after two consecutive $+1$'s; therefore a_n is fixed for n large enough and $m(x \upharpoonright n) = L$ only finitely many times. This is true for every $1 \leq L \leq \liminf m(x \upharpoonright n)$; therefore $\liminf m(x \upharpoonright n) = \infty$.

Also, for every $S \in \mathcal{B}$ and $k \in \mathbb{N}$, take L such that $(S_L, k_L) = (S, k)$. The above shows that $\inf \{\lfloor S(x \upharpoonright n) \rfloor, k\}$ is fixed for n large enough; therefore S does not oscillate around k , for every $k \in \mathbb{N}$, and since S is a V -betting strategy, it means that S does not oscillate at all. \square

Proof of Theorem 2.10. Let $\{S_1, S_2, \dots\}$ be a countable set of V -betting strategies. Assume wlog that $S_i(\sigma) \geq 0$, for every $i \geq 1$ and $\sigma \in \{-1, +1\}^{<\infty}$.

Let $\epsilon_1, \epsilon_2, \dots$ be independent random variables assume the values ± 1 with probability $(\frac{1}{2}, \frac{1}{2})$. The series $\sum_{i=1}^{\infty} \frac{\epsilon_i}{i}$ converges a.s., by Doob's martingale convergence theorem considering the boundedness of the second moment. Let $L > 0$ be so large that

$$\Pr\left(\left|\sum_{i=N}^{N+K} \frac{\epsilon_i}{i}\right| < L, \forall N, K \in \mathbb{N}\right) > 0.$$

Define a history independent \mathbb{R} -betting strategy $S_0 : \{-1, +1\}^{<\infty} \rightarrow \mathbb{R}$ as,

$$S_0(x_1, \dots, x_n) = (L + 2) \sum_{i=1}^n \frac{x_i}{i}.$$

We shall construct a random process x_1, x_2, \dots that will satisfies the following properties:

- (i) $\limsup_{n \rightarrow \infty} S_j(x \upharpoonright n) < \infty$, for $j \geq 1$ (x almost surely);
- (ii) $\liminf_{n \rightarrow \infty} S_0(x \upharpoonright n) = \infty$ (x almost surely);
- (iii) $\Pr_x(\inf_{n \in \mathbb{N}} S_0(x \upharpoonright n) \geq 1) > 0$.

Our plan is to define an increasing sequence of stopping times $n_0(x) < n_1(x) < \dots$ and to use it in a recursive definition of $\Pr(x_{n+1} | x_1, \dots, x_n)$.

Let $n_0 \equiv 0$. For $i > 0$, let

$$n_{2i-1}(x) = \inf \left\{ t > n_{2i-2}(x) : \sum_{j=1}^i S_j(x \upharpoonright t) < t \right\} \text{ and}$$

$$n_{2i}(x) = \inf \left\{ t > n_{2i-1}(x) : \sum_{n=n_{2i-1}(x)}^t \frac{x_n}{n} \geq L \right\}.$$

Now we define $\Pr(x_{n+1}|x_1, \dots, x_n)$. For $i \in \mathbb{N} \cup \{0\}$ and $n_{2i}(x) \leq n < n_{2i+1}(x)$ (here we use that n_i are stopping times),

$$\Pr(x_{n+1} = 1|x_1, \dots, x_n) \equiv \frac{1}{2}.$$

For $n_{2i+1}(x) \leq n < n_{2i+2}(x)$, the value of x_{n+1} will be determined by x_1, \dots, x_n . We write $x_{n+1} = a$ to abbreviate the expression $\Pr(x_{n+1} = a|x_1, \dots, x_n) = 1$.

Let

$$j = \inf \{e : \exists n_{2i+1}(x) \leq t \leq n, S'_e(x_1, \dots, x_t) \neq 0\}.$$

Define

$$x_{n+1} = \begin{cases} -1 & \text{if } j \leq i+1 \text{ (in particular } j \text{ is finite) and } S'_j(x_1, \dots, x_n) > 0, \\ +1 & \text{otherwise.} \end{cases}$$

We now analyze the behavior of each $S_j(x \upharpoonright n)$ by separately considering its increments over two parts of the time line: $T^{\text{odd}}(x) = \bigcup_{i=0}^{\infty} (n_{2i}(x), n_{2i+1}(x)]$ and $T^{\text{even}}(x) = \bigcup_{i=0}^{\infty} (n_{2i+1}(x), n_{2(i+1)}(x)]$.

We recursively define functions $S_j^{\text{odd}} : \{-1, +1\}^{<\infty} \rightarrow \mathbb{R}$, for $j \geq 0$, by

$$S_j^{\text{odd}}(\varepsilon) = S_j(\varepsilon),$$

$$S_j^{\text{odd}}(x \upharpoonright n) - S_j^{\text{odd}}(x \upharpoonright n-1) = \begin{cases} S_j(x \upharpoonright n) - S_j(x \upharpoonright n-1), & \text{if } n \in T^{\text{odd}}, \\ 0 & \text{if } n \in T^{\text{even}}, \end{cases}$$

and define

$$S_j^{\text{even}} = S_j - S_j^{\text{odd}}.$$

By Doob's Martingale Convergence Theorem (now we use that the S_j -s are non-negative), $S_j^{\text{odd}}(x \upharpoonright n)$ is convergent for every $j \geq 0$ (x a.s.). Also, if $n_{2i}(x)$ is finite, then so is $n_{2i+1}(x)$, (x a.s.).

For $n \in T^{\text{even}}$, the definition of x_n is so that $\{ \langle S_j(x \upharpoonright n) \rangle_{j=1}^i \}_{n=n_{2i+1}(x)}^{n_{2(i+1)}(x)}$ is non-increasing (lexicographically), it decreases each time $x_n = -1$, and it can decrease at most $n_{2i+1}(x)$ times; therefore if $n_{2i+1}(x)$ is finite, so is $n_{2(i+1)}(x)$. Also, $\langle \lfloor S_j(x \upharpoonright n) \rfloor \rangle_{j=1}^i$ is non-increasing; therefore $S_j^{\text{even}}(x \upharpoonright n)$ is eventually non-increasing (for every $j \geq 1$); and hence convergent.

By definition, $S_0^{\text{even}}(x \upharpoonright n_{2(i+1)}) - S_0^{\text{even}}(x \upharpoonright n_{2i}) \geq L > 0$. Since $\#\{n_{2i+1}(x) < n \leq n_{2(i+1)}(x) : x_n = -1\} \leq n_{2i+1}(x)$, $S_0^{\text{even}}(x \upharpoonright n) - S_0^{\text{even}}(x \upharpoonright n_{2i}) \geq -1$, for every $n_{2i}(x) < n \leq n_{2(i+1)}(x)$; therefore $\lim_{n \rightarrow \infty} S_0^{\text{even}}(x \upharpoonright n) = \infty$ and $\inf_{n \in \mathbb{N}} S_0^{\text{even}}(x \upharpoonright n) \geq -1$.

The choice of L was made so that there is an event of positive probability, \mathcal{E} , in which $\inf_{n \in \mathbb{N}} S_0^{\text{odd}}(x \upharpoonright n) \geq 2$; therefore

$$\inf_{n \in \mathbb{N}} S_0(x \upharpoonright n) = \inf_{n \in \mathbb{N}} (S_0^{\text{odd}} + S_0^{\text{even}})(x \upharpoonright n) \geq \inf_{n \in \mathbb{N}} S_0^{\text{odd}}(x \upharpoonright n) + \inf_{n \in \mathbb{N}} S_0^{\text{even}}(x \upharpoonright n) \geq 2 - 1,$$

in the event \mathcal{E} . □

Proof of Theorem 2.11. Define a history independent balanced V -strategy M by

$$\begin{aligned} M(\varepsilon) &= 2, \\ M'(\cdot \upharpoonright n) &= 1 + \frac{1}{n}. \end{aligned}$$

Denote the harmonic sum $H(n) = \sum_{k=1}^n \frac{1}{k}$. For $y \in \{-1, +1\}^{\mathbb{N}}$, we define the number of consecutive losses that M can take before going bankrupt after betting against $y \upharpoonright n$ as

$$\begin{aligned} K(y \upharpoonright n) &= k_n(M(y \upharpoonright n)), \text{ where} \\ k_n(m) &= \max \{k : k + H(n+k) - H(n) \leq m\}. \end{aligned}$$

Let $\mathcal{B} = \{S_1, S_2, \dots\}$ be a countable set of \mathbb{Z} -betting strategies. Assume wlog that $S_i(\sigma) \geq 0$, for every $i \geq 1$ and $\sigma \in \{-1, +1\}^{<\infty}$. Assume, also, without loss of generality (for technical reasons that will be clear subsequently) that \mathcal{B} includes constant strategies of arbitrary large capital.

We begin with an informal description of $x \in \{-1, +1\}^{\mathbb{N}}$. We define x recursively in two phases which are applied iteratively. In the first phase we make sure that $K(x \upharpoonright n) > S_1(x \upharpoonright n)$. In the second phase, we play adversarial to S_1 , if $S'_1(x \upharpoonright n) \neq 0$, and, otherwise, play the first phase against S_2 with initial capital $M(x \upharpoonright n) - k_n^{-1}(S_1(x \upharpoonright n))$. More generally, if at some stage n , e is maximal with respect to $K(x \upharpoonright n) > S_1(x \upharpoonright n) + \dots + S_e(x \upharpoonright n)$, we play adversarial to S_i , if $i = \inf \{j \leq e : S'_j(x \upharpoonright n) \neq 0\}$ exists, and otherwise, play the first phase against S_{e+1} with initial capital $M(x \upharpoonright n) - k_n^{-1}(S_1(x \upharpoonright n) + \dots + S_e(x \upharpoonright n))$.

We now turn to a formal recursive definition of $x \in \{-1, +1\}^{\mathbb{N}}$. Assume by induction that $x \upharpoonright n$ is already defined. Let

$$\begin{aligned} e(n) &= \inf \{e : K(x \upharpoonright n) \leq S_1(x \upharpoonright n) + \dots + S_e(x \upharpoonright n)\}, \text{ and} \\ k(n) &= K(x \upharpoonright n) - (S_1(x \upharpoonright n) + \dots + S_{e(n)-1}(x \upharpoonright n)) \end{aligned}$$

By the assumption that \mathcal{B} includes arbitrary large constants, $e(n)$ is well defined. If there is an index $1 \leq i < e(n)$ such that $S'_i(x \upharpoonright n) \neq 0$, let j be the minimum of these indexes, and

$$x_{n+1} = -\text{sign} S'_j(x \upharpoonright n).$$

Note that in this case

$$e(n+1) \geq j. \tag{3.1}$$

Assume now $S'_1(x \upharpoonright n) = \dots = S'_{e(n)-1}(x \upharpoonright n) = 0$ (or $e(n) = 1$). Define

$$x_{n+1} = \begin{cases} +1 & \text{if } S'_{e(n)}(x \upharpoonright n) \leq \frac{S_{e(n)}(x \upharpoonright n)}{k(n)}, \\ -1 & \text{if } S'_{e(n)}(x \upharpoonright n) > \frac{S_{e(n)}(x \upharpoonright n)}{k(n)}. \end{cases}$$

Note that in this case

$$e(n+1) \geq e(n). \quad (3.2)$$

Since \mathcal{B} includes constant strategies of arbitrary large capital, it is sufficient to prove that $e(n)$ diverges to ∞ and $S_i(x \upharpoonright n)$ converges for every $i \in \mathbb{N}$. Assume by negation that

$$e = \inf (\{h : S_h(x \upharpoonright n) \text{ diverges}\} \cup \{\liminf e(n)\})$$

is finite. By (3.1) and (3.2), $e(n)$ converges to e . Denote the division with remainder of $\frac{S_{e(n)}(x \upharpoonright n)}{k(n)}$ by

$$S_e(x \upharpoonright n) = q(n)k(n) + r(n).$$

For n large enough so that $e(n) = e$ and $S'_1(x \upharpoonright n) = \dots = S'_{e-1}(x \upharpoonright n) = 0$, we have $(q(n+1), r(n+1)) \leq (q(n), r(n))$ (lexicographically) with a strict inequality if either $x_{n+1} = -1$ or $k(n+1) > k(n) + 1$; therefore, for some $n_0 \in \mathbb{N}$ and every $n \geq n_0$, $x_{n+1} = +1$ and $k(n+1) = k(n) + 1$. But this is impossible since by the definition of $k(n)$ and the fact that the harmonic sum is unbounded. \square

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